

Localized eigenstates with enhanced entanglement in quantum Heisenberg spin-glasses

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The concurrence versus participation ratio phase diagram of the eigenstates of the quantum infinite range Heisenberg spin glass shows two distinct separate clouds. We show that the ‘special states’ that agglomerate away from the main one, are precisely those that are obtained by ‘promoting’ the eigenstates of a lower number sector of the Hamiltonian to particle added eigenstates of a higher number sector of the Hamiltonian. We compare the properties of these states with states of similar structure constructed from GOE random matrices that are easier to understand. In particular, we obtain the scaling behaviour of average entanglement of these special states with system size. By studying a power-law decay Hamiltonian, we see a merger of the main cloud into these special states as we go away from the infinite-range model and move towards short-range models. This could indicate that short-range quantum spin glasses are essentially different from the infinite range model.

I. INTRODUCTION

The last decade or so has seen a proliferation of interest in understanding quantum condensed matter systems from an entanglement perspective; for a somewhat early review see [1]. The notion of entanglement [2, 3] and how to measure it has developed through the aid of a number of works within the general framework of quantum information. The condensed matter systems studied have concentrated around “clean” systems that display quantum phase transitions. Exceptions include work on the von Neumann, or block, entropy in disordered models where it has been shown that the kind of scaling found in corresponding clean critical chains ($\sim \log L$, for a chain of length L) persist and may even be enhanced in the presence of quenched disorder [4].

Another somewhat different but related tack of research has been on disordered systems from the point of view of quantum chaos and random matrix theories, for example [5]. Here the applicability of measures conventionally used for few-body quantized classically chaotic systems has been sought to be applied to many-body systems without an apparent classical limit, chaotic or otherwise. The rationale being that rather than a putative classical limit, it is the nonintegrable nature of the models that determine if random matrix theories maybe applicable. Nonintegrability could naturally be found in clean systems as well and such systems display some signatures conventionally attributed to quantum chaos, for example see [6, 7]. However the well-developed theories of random matrices [8, 9] especially the Gaussian ensembles, are applicable if there are many-body interactions that tend to make the Hamiltonians full matrices rather than the typical sparse matrices that arise out of the typically two-body interactions of many-body systems. The

notion of “two-body-random ensembles” and “embedded ensembles” developed mostly within nuclear physics was precisely to plug this lacuna [10]. However its applicability to condensed matter systems is largely unexplored. It has also been pointed out that enhanced multipartite entanglement in disordered spin systems is possible that can be useful for multiport quantum dense coding [11].

The present work maybe seen in this context as an exploration of entanglement in disordered many body spin $1/2$ particles. The disorder is via the interaction that is two-body type and long-ranged. The features in these systems maybe compared to what is expected from conventional random matrix theories. Also unlike most studies related to condensed matter this work looks at not just ground states but indeed excited states as well. In particular we will study single and two-particle (or magnon) sectors. There have been many interesting studies related to quantum communication across spin chains where such subspaces have played a dominant role, see for an overview [12]. Being the simplest subspaces we also concentrate on these, although they may not contain the ground state. The kind of systems we study may then be either classified simply as long-ranged disordered Heisenberg models or quantum Heisenberg spin-glasses. Although spin glasses have been around for four decades, most studies have focussed on the classical version. See Talagrand [13] and references therein. Quantum spin glasses have also naturally been considered, for example [14]. A reason why quantum spin glasses have received less attention though is that numerical techniques to study quantum problems are less developed, as opposed to the classical world, where Monte Carlo methods are very advanced [15]. The infinite-range Sherrington-Kirkpatrick spin glass is the prototype model in classical spin glasses [16], the quantum version of which is the

starting point of our study here.

Definite-particle states are natural to quantum systems that are pure and conserve particle number or total spin. A m -particle state lies in the subspace of the Hilbert space which is spanned by the basis vectors that have m -number of ‘1’s (or equivalently ‘0’s) when expressed in spin- z basis. In one-particle states (one spin up or one spin down) there is a clear monotonic relation [17–19] between localization, for example as measured by the participation ratio, and the inter-spin entanglement as measured say by concurrence or tangle: the more the localization, the lesser the entanglement. While this has long been appreciated, the case of higher particle numbers or spins is more complicated. There is no such monotonic relationship between localization and entanglement even for two-particle states. To study this in the simplest statistical context, random definite particle states were studied using an ensemble that was uniformly distributed in such subspaces [20]. From this it is known for example that while the expected entanglement between two spins (or qubits in the language of quantum information) for one particle states having L qubits scales as $1/L$ and that of two-particle states scale as $1/L^2$, in the case of three or more particles entanglement is practically absent and is (super) exponentially small in L . This is consistent with such states having larger multipartite entanglement and the fact that entanglement moves away from being locally shared. In some sense the “environment” seen in such cases, by for example two given spins, is too large for entanglement to remain intact.

Here we make use of concurrence as a measure of inter-spin entanglement, and with the aid of numerical exact diagonalization and some analytical calculation, point out and explain a striking class of entangled states that arise from the consideration of the *quantum Heisenberg spin-glass*. Our motivation comes from trying to understand some of these striking features observed in a recent hitherto unreported study [21] of entanglement in quantum spin-glasses. In particular the two-particle sector states separate into two classes, whose entanglement scales very differently with the total number of spins L . Even those with a smaller amount of entanglement are still larger than those expected of random states. Thus these highly disordered Hamiltonians are still quite far from behaving as random states that are uniformly distributed in the definite particle subspace. Fig. 1 shows up-front two diagrams, one featuring ‘average concurrence’ and the other ‘participation ratio’ of the ‘ $N_\uparrow = 2$ ’ sector of the eigenstates of quantum Heisenberg infinite range spin glass. The illumination of the why-and-how and the consequences of these spikes observed is the core message of this paper.

The structure of the paper is as follows. In Sec. II, we describe the spin-glass Hamiltonian under study and the quantum properties of the system we are interested in. We also highlight the existence of a special class of eigenstates, which we call as ‘promoted states’. These states form the subject of this work. In Sec. III, we obtain a few

analytical results for the quantities described in Sec. II when the system is in a ‘promoted’ state, with a number of assumptions and compare with two specific spin-glass models namely the Infinite range and Nearest-Neighbour models. In Sec. IV, we show systematically that the ‘promoted’ states, that were distinguishable from the rest of the eigenstates from an entanglement perspective in the case of the Infinite range spin-glass model, become indistinguishable as the range of interaction between the spins becomes smaller. The last section summarizes the conclusions of our work, and offers an outlook for future work.

II. FORMULATION OF THE PROBLEM

A. Symmetries of the Spin-glass Hamiltonian

We start by considering a large number (L) of qubits, labelled arbitrarily from 1 to L . The generic quantum Heisenberg model is given by

$$H = \sum_{i < j} J_{ij} [\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z] = \sum_{i < j} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j \quad (1)$$

where J_{ij} represents the ‘interaction strength’ between the qubits i and j . If all the J_{ij} s are negative(positive), then the system is a ferromagnet(anti-ferromagnet). If J_{ij} s have mixed signs, then the system is said to be ‘frustrated’ and no long-range order maybe found as in [22], with a few exceptions, see for example [23].

A useful way to think of this Hamiltonian is to replace the $\vec{\sigma}_i \cdot \vec{\sigma}_j$ with $2\hat{S}_{ij} - 1$, where \hat{S}_{ij} is the swap operator that interchanges the z -component of the spins of qubits i and j . It is immediately clear that the Hamiltonian connects only states of the same particle number (number of spin-up qubits). Thus, when expressed in the σ_z basis (or Fock state basis), the Hamiltonian takes a block diagonal form, where each block is characterized by a definite “particle number” N_\uparrow , which is nothing but the total number of spins up in the z direction. This is a great simplification since we can study only one or a few definite particle sectors at a time and the associated Hilbert space \mathcal{H}_{N_\uparrow} is much smaller and grows only polynomially with L , the number of qubits. This definite particle structure is of course a manifestation of the fact that the system can have any of the $L + 1$ allowed values for the total z -spin, but once a value is assumed, it is conserved until acted by a random external magnetic field. Mathematically, it is the consequence of rotational symmetry and in particular, of the fact that $\sigma_z = \sum_i \sigma_i^z$ commutes with the Hamiltonian. Considering the isotropy of the Hamiltonian, it also follows that the operators $\sigma^\pm = \sum_i \sigma_i^\pm$ too commute with the Hamiltonian. This implies that if $|\psi\rangle$ is an eigenstate of H , then $\sigma^\pm |\psi\rangle$ must also be an eigenstate of H with the same eigenvalue. The particle added state $\sigma^+ |\psi\rangle$ is referred to as a “promoted” state corresponding to $|\psi\rangle$.

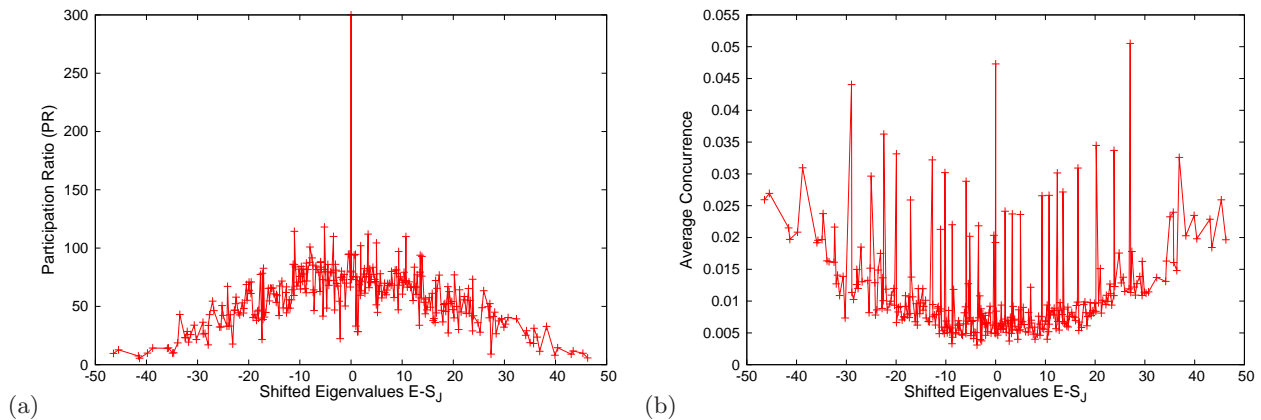


FIG. 1. A typical realization of the Hamiltonian for $L = 25$, $N_\uparrow = 2$. We plot the participation ratio and average concurrence (averaged over all pairs of spins chosen from the L spins) of all the eigenstates of the infinite range quantum Heisenberg spin glass arranged by their eigenvalues. S_J refers to $\sum_{i,j} J_{ij}$, the energy of the all-one state.

For any definite-particle sector of the Hamiltonian, one can show (readily from the swap operator form) that the state with equal coefficients for all the basis states is necessarily an eigenstate, with eigenvalue $S_J := \sum_{i < j} J_{ij}$. This corresponds to states promoted progressively from the 0-particle state $|0\rangle^{\otimes L}$ and we will refer to such states as ‘all-one’ states of the appropriate particle number.

The two models of spin-glass that we use intensively in this work are

1. Infinite-range model where $J_{ij} \sim \mathcal{N}(0, 1) \forall i < j$.
2. Nearest-Neighbour model where $J_{ij} \sim \mathcal{N}(0, 1)$ iff $j = i - 1$ else 0, with a periodic condition such that $j = 0$ corresponds to $j = L$.

Here, $\mathcal{N}(0, 1)$ stands for the standard normal distribution. Also, we briefly study a one-dimensional power law decay model that contains the above two models as special cases.

B. Entanglement & Localization

As stated in the introduction, for 1-particle states, there is a direct relation between the localization of a state and the average entanglement between qubits [17–19] while no definite relation is known to exist in the case of higher particle states. We will be interested in concurrence as a measure of bipartite entanglement and participation ratio (PR) as a measure of localization.

Concurrence is the entanglement between any two two-level systems in a mixed or pure state. Therefore it is a good measure of entanglement between any two qubits or spin-1/2 particles in a general many-body state. For definite particle states, it is known that the reduced density

matrix of any two spins takes the following form [24]:

$$\rho = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & w & z & 0 \\ 0 & z^* & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix} \quad (2)$$

With ρ of such a form, concurrence has a simple expression [24]:

$$C(\rho) = \max(2(|z| - \sqrt{vy}), 0). \quad (3)$$

Appendix describes a simple algorithm to do a fast computation of C for Hamiltonians with this symmetry.

Next, we recall that the inverse participation ratio is a basis dependent quantity that is defined as follows. If a state $|\psi\rangle = \sum_i a_i |i\rangle$, where $|i\rangle$ are the kets in the computational basis, then $\text{IPR} = \sum_i a_i^4$, from which the participation ratio $\text{PR} = \frac{1}{\text{IPR}}$ is immediately obtained. It is typical to consider the S_z basis for spin-systems, and we do the same.

During our study of entanglement in the spin-glass systems [21], in addition to the high-PR-high-concurrence all-one state, several other states with high concurrence were observed. But these states did not stand out when a similar plot of PR was made. Refer Fig. 1. It is clear that the eigenstates would clearly separate out into two distinct clouds in a phase diagram where we plot the average concurrence against PR (+ points in Fig. 2). The spikes at $E - S_J = 0$ in both Fig. 1 (a) and (b) correspond to the 2-particle all-one state. While no other significant spikes are found in Fig. 1(a), several such spikes are found in Fig. 1(b), with some of them being greater than the spike for all-one state. The authors were motivated by this observation to investigate further, which resulted in this work.

Note that the 2-particle all-one state is not the state with the highest average concurrence, while the 1-particle all-one state, with pairwise concurrence between any two qubits being $2/L$, has the maximum possible average pairwise entanglement [17]. Next, we show

that these spikes are in fact the set of all ‘promoted’ eigenstates, arising due to the isotropy of the Hamiltonian.

III. PROMOTED STATES

Recollect that promoted states are of the form $\sigma^+ |\psi\rangle$. It is easy to show that the operator $\sigma^+ \sigma^-$ leaves the promoted states invariant, upto a scale factor. We mark those states that have this property in Fig 2 with boxes. Also we include data for random states and ‘promoted’ random states.

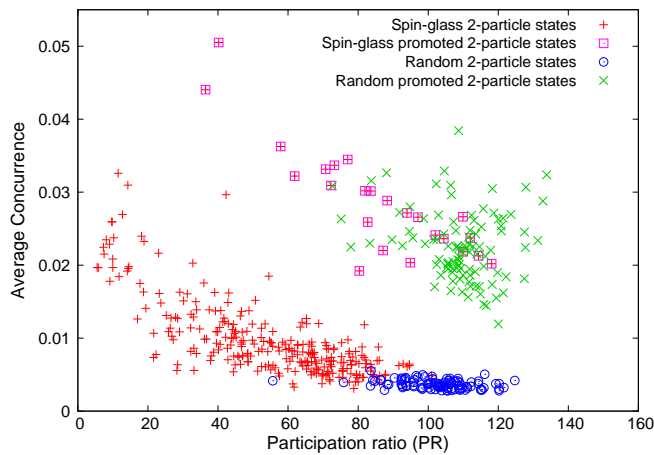


FIG. 2. Scatter plot the average concurrence vs. the participation ratio (PR) of all the eigenstates of the $N_\uparrow = 2$ sector in infinite range quantum Heisenberg spin glass for $L = 25$. Data for *random* states and *random-promoted* states are included.

It is known [20] that concurrence of a random 2-particle state ($N_\uparrow = 2$) is given by $\frac{16}{L^2 \pi^{3/2}}$, which with $L = 25$ gives ~ 0.005 . We infer that the eigenstates of these Hamiltonians tend to have higher concurrence and are more localized than random states (circles). Clearly, the distribution of points for the eigenstates arising from the Hamiltonian is very different from what is expected for random states. This immediately suggests that entanglement properties of random states, such as scaling behavior [20] may not be found for eigenstates of spin systems, at least those with two-body interactions [21].

The most striking feature in the infinite-range model is that the eigenstates cluster into two separate clouds. From Fig. 2, we see that it is the promoted states, identified and labelled as boxes (over + points), that stand out and form a separate cloud. These promoted states also tend to be distinct when we consider 2-qubit and 3-qubit entanglement in 3-particle sector as we verified (figure not included).

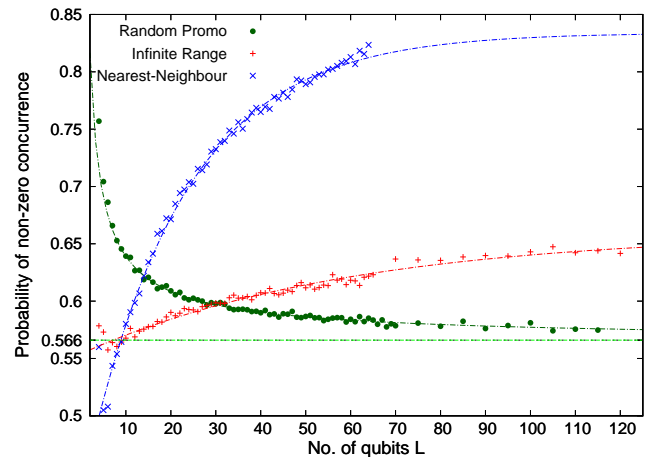


FIG. 3. $P(C > 0)$ plotted against L . The dashed lines are the fits taking into account of these errorbars and the dotted-dashed lines are fits without taking the errorbars into account. Points with $L < 8$ are not considered for finding fit parameters.

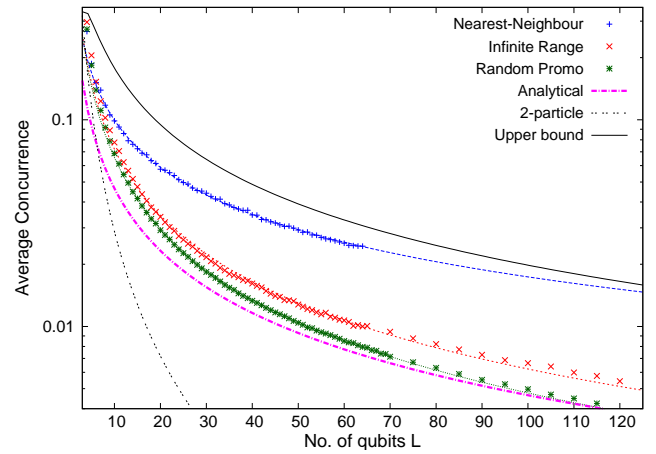


FIG. 4. Scaling behaviour of the promoted states compared against the scaling behaviour expected for random 2-particle states, which is $\frac{16}{L^2 \pi^{3/2}}$ and random promoted 2-particle states and its estimate. ‘Analytical’ refers to curve $0.465/L$. The solid line corresponds to the upper bound given in Sec. III B. The fit parameters are given in Table III. The y-axis is in logscale for better visibility.

A. Random Promoted States

The correlations in the matrix elements and the structure of the Hamiltonian make any analytical study of the spin-glass eigenstates complicated. In order to understand why the ‘promoted’ nature makes states special, we study *random* promoted states which are amenable to an analytical approach. From the orthogonality condition applied with respect to the ‘all-one’ eigenstate, for all other eigenstates, the coefficients in the standard basis must add to zero. We will enforce this property extensively on the random promoted states to simplify our ex-

pressions. Random 1-particle states, by definition, span the surface of the unit sphere in the associated Hilbert space uniformly. In addition, we impose the constraint that the states must lie in the hyperplane orthogonal to the all-one state, to mimic the promoted states arising in spin-glass systems. Thus, if we denote an unnormalized random 1-particle state as

$$|\psi_{1p}\rangle = \sum_i a'_i |i\rangle, \quad (4)$$

where $|i\rangle$ refers to the 1-particle basis state where only the qubit at i is in $|1\rangle$ state and the rest in $|0\rangle$, then the a'_i s come from a j.p.d.f containing $\delta(\sum_i a'_i)$ but we shall still consider them as i.i.d variables for all practical purposes. Isotropy in the Hilbert space is achieved when their marginals happen to be the standard normal distribution $\mathcal{N}(0, 1)$, see for example [25]. The normalization is assumed to only set the scale of the coefficients and the normalized coefficients can continued to be treated independent of each other [20]. Although we restrict ourselves to the subspace that satisfies $\sum_i a_i = 0$ for theoretical analyses that follow, in practice, we do not include this condition in the Monte Carlo simulations.

The action of the ‘promotion-operator’, σ^+ , on a 1-particle state is given by

$$\sum_k a_k |k\rangle \xrightarrow{\sigma^+} \sum_{i<j} a_{ij} |ij\rangle, \text{ where } a_{ij} \sim a_i + a_j. \quad (5)$$

Here, $|ij\rangle$ refers to the 2-particle basis state where the qubits at i and j are in $|1\rangle$ state and rest in $|0\rangle$. The IPR of the promoted 2-particle state can be expressed in terms of the IPR of the corresponding 1-particle state when $\sum_i a_i = 0$:

$$\sum_{i<j} a_{ij}^4 = \frac{1}{(L-2)^2} \left((L-8) \sum_i a_i^4 + 3 \right), \quad (6)$$

where $a_{ij} = (a_i + a_j)/\sqrt{L-2}$, are the normalized coefficients. In finding the normalization, the condition $\sum_i a_i = 0$ is used, which is the case for eigenstates of spin-glass Hamiltonian. In a more general case, the normalization depends on the coefficients themselves. We observe in passing the somewhat amusing fact that whatever maybe the 1-particle state, when $L = 8$ the promoted 2-particle state has an IPR of exactly $1/12$, provided the coefficients sum to zero.

For random N dimensional states, the average IPR $\langle I \rangle \sim 3/N$ [26]. Using this in Eq. 6 for random 2-particle promoted states, the average IPR

$$\langle I_{\text{promoted 2p}} \rangle \sim \frac{6}{L^2}, \quad (7)$$

same as what we expect for ‘genuine’ 2-particle random states, as the dimensionality of such a space is $\sim L^2/2$, and is confirmed by Fig. 2. Thus as far as localization is

concerned there is no difference, typically, between promoted and genuine 2-particle states. We find a very different situation regarding quantum correlations such as entanglement to which we now turn.

The elements of the two-spin reduced density matrix that are involved in the entanglement between them, as quantified by concurrence, are z, v, y (refer to Eqs. 2 and 3). For 2-particle states, when ρ is the density matrix of spins at positions 1 and 2 (which we consider for simplicity and without any loss of generality), these elements are

$$y = a_{12}^2, \quad z = \sum_{k=3}^L a_{2k} a_{1k}, \quad v = \sum_{\substack{k,l=3 \\ k<l}}^L a_{kl}^2. \quad (8)$$

Note that we are considering *real* state ensembles. We may expect that while all the three quantities are random variables, y being a single term has more fluctuation than the other two which are sums. For generic random 2-particle states $\langle |z|^2 \rangle \sim 4/L^3$ and $\langle |z| \rangle^2 = (2/\pi) \langle |z|^2 \rangle$ [20].

However for 2-particle states promoted from 1-particle states obeying $\sum_i a_i = 0$, it is straightforward to show that, up to the leading order,

$$y \approx (a_1 + a_2)^2/L, \quad z \approx (1 + La_1 a_2)/L, \quad v \approx 1. \quad (9)$$

Therefore although z appears as a sum of order L number of terms, it simplifies for promoted states to this simple form which implies that both $|z|$ and \sqrt{y} are of the same order of magnitude, namely $1/L$. This follows since $a_i \sim 1/\sqrt{L}$. As $v = \mathcal{O}(1)$, the concurrence in promoted 2-particle states is always in a fine balance between the two competing terms $|z|$ and \sqrt{y} . In contrast, for generic 2-particle states, the order of \sqrt{y} is $1/L$ which is much larger than the order of $|z|$ which is $1/L^{3/2}$, resulting in the probability that the concurrence is nonzero decreasing as $1/\sqrt{L}$ [20].

The probability that the concurrence is nonzero for promoted 2-particle states is now estimated. This is the same as $P(z^2 > y)$, which using Eq. (9) results in the following where $x_i = \sqrt{L} a_i$ are independently distributed according to $\mathcal{N}(0, 1)$.

$$P(C > 0) = P[(1 - x_1^2)(1 - x_2^2) > 0] = \text{erf}^2\left(\frac{1}{\sqrt{2}}\right) + \text{erfc}^2\left(\frac{1}{\sqrt{2}}\right) \approx 0.566. \quad (10)$$

Note that as $v < 1$ in reality, the above can be expected to underestimate the actual probability. It is also worth recounting that random 1-particle states have a probability 1 that the concurrence is nonzero.

Fig. 3 shows how $P(C > 0)$ varies with L for different systems. We fit a model of the form $p + q/L^r$ to the data points from Monte Carlo simulations for random promoted states (without enforcing $\sum_i a_i = 0$), and the values for the parameters obtained are tabulated in Table I.

Parameter	Estimate
p	0.564 ± 0.002
q	0.426 ± 0.032
r	0.754 ± 0.042

TABLE I. Best fit curve $p + q/L^r$ to $P(C > 0)$ vs. L for random promoted 2-particle states. Fits have been made for $L \geq 8$.

The probability of non-zero concurrence approaches from above the theoretical asymptotic value of 0.566 slower than $1/L$. In sharp contrast, the probability of non-zero concurrence *increases* with L for the spin-glass systems, with the nearest-neighbour model increasing faster than the infinite-range model. The $P(C > 0)$ curve is described an exponential curve of the form $p - q \exp(-L/r)$. The functional forms are purely data driven and are not motivated by any physical argument.

Parameter	Infinite range	Nearest-Neighbour
p	0.660 ± 0.004	0.834 ± 0.003
q	0.106 ± 0.003	0.402 ± 0.004
r	59.565 ± 5.365	21.891 ± 0.585

TABLE II. Best fit curves $p - q \exp(-L/r)$ to $P(C > 0)$ vs. L for promoted 2-particle states of Infinite range and Nearest-Neighbour spin-glass models. Fits have been made for $L \geq 8$.

The average concurrence of random promoted 2-particle states may also be estimated as

$$\langle C \rangle \approx 2 \int_{\prod_i (1-x_i^2) > 0} (|z| - \sqrt{vy}) e^{-(x_1^2+x_2^2)/2} dx_1 dx_2 \quad (11)$$

$$\approx \frac{0.465}{L},$$

where $x_i = \sqrt{L}a_i$ and Eq. 9 and the assumption of independent marginals have been used. The final result was obtained by setting $v = 1$ and factoring out the L dependence and the L -independent integral was evaluated numerically to obtain 0.465. The $1/L$ behaviour is to be compared with generic 2-particle states that have an average concurrence $\sim 1/L^2$ [20], and that for generic random 1-particle states which goes as $\sim 1/L$ [17]. The promoted states have a smaller entanglement than 1-particle states, however they are much larger than what maybe expected for generic 2-particle states.

Interestingly, for the promoted 0-particle state i.e. the all-one state in the 1-particle ($N_\uparrow=1$) sector, the concurrence between any two spins is $2/L$ and in 2-particle ($N_\uparrow=2$) sector, it is

$$C = \frac{2}{\binom{L}{2}} \left(L - 2 - \sqrt{\frac{(L^2 - 5L + 6)}{2}} \right), \quad (12)$$

which also scales as $1/L$.

Fig. 4 shows how in our Monte Carlo simulations the average concurrence scales with the size for different systems. The exponent is smaller for the promoted 2-particle eigenstates when compared to promoted 2-particle random states. We fit a model $\langle C \rangle = b/L^a$ to the points ($L \geq 8$) and obtain parameters displayed in Table III.

Model	a	b
Infinite Range	1.063 ± 0.008	0.832 ± 0.023
Nearest-Neighbour	0.758 ± 0.003	0.570 ± 0.006
Random Promo	1.138 ± 0.004	0.900 ± 0.014

TABLE III. Best fit curve b/L^a to $\langle C \rangle$ vs L . Fits have been made for $L \geq 8$.

Thus the promoted random 2-particle states retain the larger entanglement present in 1-particle states while being delocalized like 2-particle states. It is interesting that this feature of random states is present intact in the eigenstates of quantum spin glass Hamiltonians with long range interactions.

B. Heuristic understanding of the deviations

From Fig 4, it is clear that the infinite range spin-glass Hamiltonian and nearest-neighbour (NN) Hamiltonian differ from each other and from the random promoted states. The average entanglement in the spin-glass eigenstates is higher than the entanglement in random states, which can be related to the higher probability of non-zero concurrence. We can understand this as follows: In the case of 1-particle sector, eigenstates of the Hamiltonian are typically more localized than random states and as a result, less entangled. When promoted to the 2-particle sector, these states tend to be more entangled than the random promoted states.

Random states may be viewed as the ensemble of eigenstates of random matrices that belong to the Gaussian Orthogonal Ensemble (GOE). GOE matrices are symmetric, with off-diagonal terms being i.i.d and drawn from $\mathcal{N}(0, 1)$ and the diagonal terms drawn from $\mathcal{N}(0, 2)$, where as for the spin-glass Hamiltonians under consideration, the variance of the diagonal terms is much larger when compared to that of the off-diagonal terms. The variance of the diagonal terms in the 1-particle sector of the spin-glass Hamiltonian matrices are $\sim L^2/2$ times bigger than that of the non-zero off-diagonal elements. As a result, the diagonal elements end up being typically much larger in magnitude when compared to the off-diagonal terms and hence are ‘close’ to being a diagonal matrix, with the Nearest-Neighbour model being ‘closer’ because of its sparse structure.

For a diagonal matrix, the basis vectors are its eigenvectors (in the non-degenerate case). Thus, we expect the energy eigenstates of the spin-glass systems to be typically localized. As a toy problem, consider a 1-particle

basis state i.e. where only one of the coefficients is non-zero and hence 1 (upto a sign). The condition $\sum_i a_i = 0$ is no longer applicable and the corresponding promoted state is given by $a_{ij} = (a_i + a_j)/\sqrt{L-1}$.

We can factor out the state of the qubit with up spin, implying that that particular qubit is not entangled to the remaining $L-1$ qubits, that are in the maximally entangled (on average) all-one 1-particle state. Given such a promoted localized state, the probability of non-zero concurrence is $\binom{L-1}{2}/\binom{L}{2} \sim 1$ and the concurrence between any two pair of qubits, when not zero is $2/(L-1)$. Thus, the average concurrence $\langle C \rangle = \frac{2}{L-1} \frac{L-2}{L}$. This is plotted as the solid line in Fig. 4

However, in the Nearest-Neighbour model, for large values of L , the diagonal elements tend to assume the eigenvalue corresponding to the all-one state. This leads to a high degeneracy and as a result, we have a high dimensional eigen sub-space corresponding to the eigenvalue $\sum_{i<j} J_{ij}$. Since the quantities of interest are basis dependent, it is no longer meaningful to discuss about the average concurrence and participation ratio of the eigenstates for large values of L .

IV. THE ONE-DIMENSIONAL POWER-LAW DECAY MODEL

To study the differences between the infinite-range and nearest-neighbour models, let us introduce a notion of distance that have been absent so far. Consider a family of Hamiltonians with the spins being put on a closed chain and where the interactions fall with distance as a power law, with periodic boundary conditions imposed. The distance between the i^{th} and j^{th} spins is taken to be the length of the chord between the two sites, when all the sites are put on a circle [27] given by

$$r_{ij} = \frac{L}{\pi} \sin \left[\frac{\pi}{L} (i - j) \right]. \quad (13)$$

The J_{ij} s defined in Eq. 1 obey $\mathcal{N}(0, 1/r_{ij}^\sigma)$. We obtain the infinite range Heisenberg spin-glass when $\sigma = 0$ and recover the 1-dimensional nearest neighbour model asymptotically as $\sigma \rightarrow \infty$.

The cloud of non-promoted eigenstates has a tendency to merge with that of promoted states as we increase σ as shown in Fig 5. In the study of the same model with classical spins [27], the regime $\sigma = (0, 0.5)$ was identified as the infinite-range universality class, the regime $\sigma = (0.5, 0.67)$ as the mean-field universality class and $\sigma = (0.67, 1)$ as short-range. So, in that case anything greater than $\sigma = 1$ is already in the super-short range regime, and the system should already have characteristics of the nearest-neighbor model. But in the quantum version of the model, even at $\sigma = 2$, a faint separation of the clouds can still be discerned, and only at $\sigma \sim 2.5$, the figure is for all practical purposes, is same as what we obtain for nearest neighbour spin chain. Here we note that

it is the lower cloud that merges into the cloud of promoted states. The position of the promoted eigenstates in the Concurrence-PR plot is relatively stable. Thus, the properties of these states are relatively robust and must not be too sensitive to our assumption of the model. This could be advantageous in some eventual quantum information application, since defects in physical realization of spin glass wouldn't affect these states much.

There are two interesting aspects that come out from the phase diagram for large σ (short-range). One is that apparently there are no distinct clouds. The second is that inspite of the absence of a clear separation between clouds, the 'special eigenstates' continue to occupy the higher concurrence - higher PR region. A purely numerical approach can perhaps never really tell whether the merger is a true merger or if the clouds continue to exist, but only get arbitrarily close.

V. CONCLUSIONS AND FUTURE WORK

We have discovered that in the general spin-glass Hamiltonian, a special class of eigenstates form a distinct cloud in the concurrence-PR phase diagram. These special eigenstates, which we show to be 'promoted-eigenstates', display significantly higher concurrence compared with the rest of the eigenstates. As a first step to understand the peculiarities of 'promoted-eigenstates', we have studied the properties of random promoted states by an analytical approach. We find that the random promoted states are quite different from the promoted eigenstates of the Hamiltonian, which we attribute to the additional structures in the Hamiltonian. While we understand some deviations qualitatively, a lot of them still remain a mystery.

By considering a power-law decay model in 1-d, we have shown that the 'regular eigenstates' tend to merge with the 'special eigenstates' as we move further and further away from the infinite-range case and move towards the short-range model. This is a significant observation because it could indicate that there is something inherently different between the infinite-range and short-range models in the quantum version of the model. If one speculates that there is some direct connection between the quantum and classical version of the Heisenberg spin glass, this could indicate that perhaps the RSB picture which is valid for the SK model, may not be applicable for the short-range models. The application of a uniform magnetic field to the quantum spin-glass, in spite of breaking the symmetry with regard to the σ^\pm operators, preserves the set of eigenstates of the zero-field Hamiltonian, and therefore the promoted eigenstates continue to exist. However, if the applied external magnetic field is random over different sites, there would be no special promoted eigenstates. Some of our preliminary checks on the infinite-range Hamiltonian suggest that in fact, the clouds remain in tact in the presence of a uniform field, whereas they disappear with random

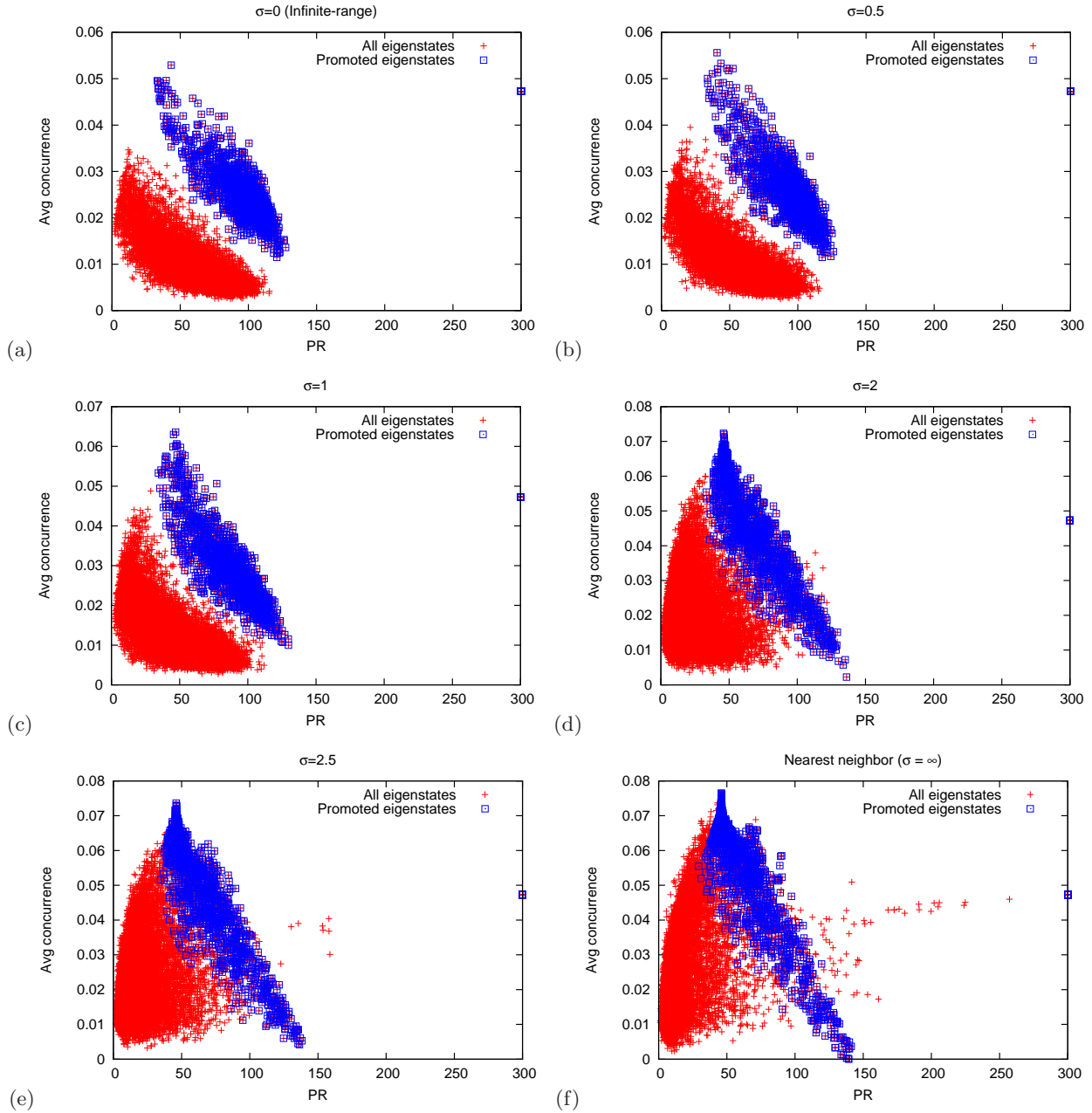


FIG. 5. Scatter plot of the average concurrence vs. participation ratio PR of all the $N_{\uparrow} = 2$ eigenstates of the 1-d power-law decay Heisenberg spin glass for $L = 25$, with periodic boundary conditions for $N_{\text{samp}} = 50$ samples of disorder. We show data for $\sigma = 0$ (infinite-ranged), 0.5, 1, 2, 2.5, and ∞ (nearest-neighbour). Notice that $\sigma = 2.5$ looks for all practical purposes like the strict ‘nearest-neighbour’. Every eigenstate is marked as +. Promoted eigenstates also marked as boxes.

fields. In a recent work [28], it was shown that the so called Almeida-Thouless line of phase transitions exists for the infinite-range *classical* vector spin glass under the application of *random* fields. It would therefore be exciting to investigate if and what consequence the disappearance of these special promoted eigenstates has to the Almeida-Thouless line in *quantum* Heisenberg spin glasses. The AT line lies at the heart of the RSB-versus-droplet-picture debate in classical spin-glasses, and it would be a significant avenue of research to study the

same with quantum spins. Another future work [21] of interest for us involves the study of the impact of these promoted states in inhibiting the transition from power-law to exponential scaling as reported in Vijayaraghavan et al [20].

We end this paper by suggesting an experimental technique to attain these promoted m -particle states for applications that may require localized yet highly entangled states. Starting from the 0-particle eigenstate, one can adiabatically flip any one qubit and the system

would be in the all-one state of the 1-particle sector. Now holding the particle number constant, we can ‘heat up’ or ‘cool down’ the system so that it is in any other 1-particle state. Now again, we could adiabatically flip one of the qubits and obtain a promoted 2-particle state. By repeating the process of adiabatic flipping and heating/cooling, we can obtain a promoted m -particle state.

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package [29]. Curve fitting to find optimal parameters, including the errorbars, were done using SciPy’s `curve_fit` routine.

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Appendix: Algorithm for concurrence

Computing the average concurrence of a given state turns out to be computationally the most expensive part of our code¹. Here we give a brief description of the algorithm we used to compute the essential elements of the reduced-density-matrix-of-any-two-spins ρ en-route to computing the concurrence for systems with definite-particle-symmetry for any particle number.

Let us say we are interested in the concurrence between sites i, j of some particular eigenstate $|\psi\rangle = \sum_k a_k |k\rangle$, where $|k\rangle$ are the basis states. For a given pair i, j , we first initialize all the elements of the reduced density matrix (which is real in our case) to 0 and then loop over all k . For each state $|k\rangle$, we first find the states of the i and j sites. If both i and j have up(down) spins we add a_k^2 to $v(y)$. The off-diagonal element is a bit more complicated and requires another loop and a search algorithm within, but not difficult. If i has an up(down) spin and j has a down(up) spin, we do the following. We find all states in the basis which have a *down(up)* spin at i and an *up(down)* spin at j , but are exactly identical to the state $|k\rangle$ at every other site. Let us call such states $|l\rangle$, which thus have coefficients a_l in the eigenstate $|\psi\rangle$ of interest. We just add a_k^2 to $w(x)$ and add sum of the products

¹ <https://github.com/arunkannawadi/spinglass>

$a_k a_l$ for all l to z . Although, w and x are not required to compute concurrence, they may be required for other measures of entanglement, such as log-negativity or the Von-Neumann entropy that measures how the qubits i and j are entangled to rest of the system. Having thus computed v, y , and z , the concurrence is trivially obtained by the formula of Connor and Wootters [24].

For the 2-particle states, this algorithm loops over all possible pairs of spins ($\sim \mathcal{O}(L^2)$), over all basis vectors ($\sim \mathcal{O}(L^2)$) and for $(L-2)$ cases out of $\binom{L}{2}$, it compares the k^{th} basis with all other basis, which goes as $\mathcal{O}(L^2)$. In short, calculating the average concurrence for any given 2-particle state is $\mathcal{O}(L^5)$. A full analysis of a 2-particle states will involve diagonalization of $N_{\uparrow}=2$ sector of the Hamiltonian which is $\mathcal{O}(L^6)$. But it would be wise to

obtain the eigenstates of $N_{\uparrow}=1$ sector of the Hamiltonian, which is $\mathcal{O}(L^3)$, and then ‘promote’ them to 2-particle states. One might also consider computing concurrence between randomly selected pairs of spins instead of all spins.

The most compact way to represent the basis vectors is to denote them by the positions of up spins, as denoted by $|ij\rangle$. But we quickly realized that generalizing it to an arbitrary definite-particle state is tedious. Instead, the basis are represented by integers and the binary strings of the basis vectors are encoded in bit representation of integers. In order to be able to use large values of L , the 128 bit integers in `C++ Boost`² libraries were used. This is generic and any m -particle basis vectors can be easily created.

² <http://www.boost.org>